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HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

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THE MODERN EXPONENTIAL NOTATION
(*Continued*).

J. H. Rahn's *Deutsche Algebra*, printed in Zürich, 1659, contains for positive integral powers two notations, one using the cartesian exponents, a^3, x^4, the other consisting in writing a small spiral between the base and the exponent on the right. Thus a ɔ 3 signifies a^3. The spiral signifies *involution,* a process which he calls *involviren.* An English translation, altered and augmented by John Pell, was made by Thomas Brancker and published 1668 in London.[1] In the same year positive integral exponents were used by Lord Brouncker in an early volume of the *Philosophical Transactions* of London.[2] In these transactions none of the pre-descartian notations for powers appear, except a few times in an article of 1714, written by John Cotes.

Of interest is the following passage in Newton's *Universal Arithmetick* (which consists of lectures delivered at Cambridge in the period, 1669–1685 and first printed 1707): "Thus $\surd 64$ denotes 8; and $\surd 3 : 64$ denotes 4. . . . There are some, that to denote the Square or first Power, make use of q, and of c for the Cube, qq for the Biquadrate, and qc for the Quadrato-Cube, etc. . . . Others make use of other sorts of Notes, but they are now almost out of Fashion."[3] In an edition of 1679 of the works of Fermat the algebraic notation of Vieta, originally followed by Fermat, is discarded in favor of the exponents of Descartes.[4] It would seem, from what has been cited, that about 1660 or 1670 the positive integral exponent had won an undisputed place in algebraic notation. Though generally adopted, it was not universally so. A large volume, the P. Gasparis Schotti *Cursus mathematicus,* Frankfurt a. M., 1661, and the second edition of Diophantus by Bachet de Méziriac (Toulouse, 1670), contain no trace of the modern exponential notation. Joseph Raphson's explanation of his method

[1] See G. Wertheim in *Bibliotheca mathematica,* 3d S., Vol. III, 1902, pp. 113–126.

[2] *Phil. Trans.*, Vol. III, for anno 1668, printed 1669, p. 647.

[3] Newton's *Universal Arithmetick,* London, 1728, p. 7.

[4] *Œuvres de Fermat.* Éd. PAUL TANNERY ET CHARLES HENRY, T. I, Paris, 1891, p. 91 foot-note.

of approximation to the roots of numerical equations, printed in the Latin edition of John Wallis's *Algebra,* in 1693, does not use positive integral exponents; Raphson uses powers of g up to g^{10}, but in every instance he writes out each of the factors, after the manner of Harriot.

It is worthy of note that for a long while there were two different notations for the *square* of a letter. Some wrote aa; others a^2. It would be rather difficult to make out a clear case in favor of a^2, were one to base the argument on the greater economy of space. The symbolism aa was preferred by Descartes, Huygens, Rahn, Kersey, Wallis, Newton, Halley, Rolle, L. Euler — in fact, by most writers of the second half of the seventeenth and of the eighteenth centuries; a^2 was preferred by Leibniz, Ozanam, David Gregory.

Negative and fractional exponential notations had been suggested by Chuquet, Stevin and others. The modern symbolism is due to Wallis and Newton.

In his *Arithmetica infinitorum,* Oxford, 1656, Wallis uses positive integral exponents and speaks of negative and fractional "indices."[1] But he does not actually write a^{-1} for $1/a$, or $a^{\frac{3}{2}}$ for $\sqrt{a^3}$. He speaks of the series $1/\sqrt{1}$, $1/\sqrt{2}$, $1/\sqrt{3}$, etc., as having the "index $-\frac{1}{2}$," the series 1, 4, 9, $\cdots$ as having the "index 2," the series $\sqrt{1}$, $\sqrt{8}$, $\sqrt{27}$, $\cdots$ as having the "index $\frac{3}{2}$."[2] Our modern notation involving fractional and negative exponents was formally proposed about a dozen years later. On June 13, 1676, Newton wrote to H. Oldenburg, then secretary of the Royal Society of London, a letter which was forwarded to Leibniz. The letter contains the following passage, which is interesting as containing the binomial theorem and explaining the use of negative and fractional exponents:

Sed extractiones radicum multum abbreviantur per hoc Theorema.

$$\overline{P+PQ}|^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

Ubi $P + PQ$ significat quantitatem, cujus radix, vel etiam dimensio quævis, vel radix dimensionis, investiganda est; P, primum terminum quantitatis ejus; Q, reliquos terminos divisos per primum. Et m/n, numeralem indicem dimensionis ipsius $P + PQ$: sive dimensio illa integra sit; sive, ut ita loquar, fracta; sive affirmativa, sive negativa. Nam, sicut analystæ, pro aa, aaa, &c. scribere solent a^2, a^3, &c. sic ego, pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{ca^5}$, &c. scribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$; & pro $1/a$, $1/aa$, $1/aaa$, scribo a^{-1}, a^{-2}, a^{-3}.[3]

[1] J. Wallis, *Arithmetica infinitorum,* 1656, p. 80, Prop. CVI.

[2] Of interest is the following quotation from a discussion by T. P. Nunn, in the *Mathematical Gazette,* Vol. VI, 1912, p. 255: "Those who are acquainted with the work of John Wallis will remember that he invented negative and fractional indices in the course of an investigation into methods of evaluating areas, etc. He had discovered that if the ordinates of a curve follow the law $y = kx^n$, its area follows the law $A = 1/(n+1) \cdot kx^{n+1}$, n being (necessarily) a positive integer. This law is so remarkably simple and so powerful as a method that Wallis was prompted to inquire whether cases in which the ordinates follow such laws as $y = k/x^n$, $y = k\sqrt[n]{x}$ could not be brought within its scope. He found that this extension of the law would be possible if k/x^n could be written kx^{-n}, and $k\sqrt[n]{x}$ as $kx^{1/n}$. From this, from numerous other historical instances and from general psychological observation, I draw the conclusion that extensions of notation should be taught because and when they are needed for the attainment of some practical purpose, and that logical criticism should come *after* the suggestion of an extension to assure us of its validity."

[3] Isaaci Newtoni *Opera* (ed. S. Horsley), Tom. IV, Londini, 1782, p. 525.

It is worthy of note that the modern fractional exponent was first introduced by Newton in the announcement of his Binomial Theorem, invented by him some time before 1669. Newton used also negative fractional exponents. In November, 1676, Leibniz collected some of his results on a sheet of paper; he uses here the notation[1] x^{-3}, $x^{-\frac{1}{2}}$. It should be observed also that the use of *literal* exponents is suggested in Newton's form for the binomial theorem as given above, and that literal exponents came to be generally used in the latter part of the seventeenth century by Newton, Leibniz and their followers. Perhaps the earliest occurrence of literal exponents is in Wallis's *Mathesis universalis*, Oxford, 1657, where a few expressions like $\sqrt{}\,{}^{d}R^{d} = R$, $AR^{m} \times AR^{n} = A^{2}R^{m+n}$ have been noticed.[2]

The theory of exponents, involving positive, negative and fractional values, is explained and freely used in the much respected and widely read work, entitled, *Analyse demontrée*, by Charles Reyneau, Paris, 1708. The theory is explained in the introduction to the first volume. This is done because the treatises on algebra then in use did not usually contain it. Reyneau uses these words: "Le seul calcul qui n'est pas expliqué dans les Traités d'Algebre dont on vient de parler, est celui des exposants des puissances."[3] In deriving rules for differentiation,[4] Reyneau passes from $x^{x} = a$ to $xlx = la$, and from $x^{x^{x}} = y^{y^{y}}$ to $x^{x}lx = y^{y}ly$.

The interesting question arises, when and where did the union between the exponential and logarithmic concepts take place? It did not occur until the eighteenth century. As is quite proper, there was quite a long courtship. It goes back to the time of Wallis. In the twelfth chapter of his *Algebra*, 1685, Wallis develops the theory of logarithms, beginning with two progressions 1, 2, 4, 8, $\cdots$ and 0, 1, 2, 3, $\cdots$. He then generalizes by taking

$$\begin{array}{l} 1 \cdot r \cdot rr \cdot r^{3} \cdot r^{4} \cdot r^{5} \cdot r^{6} \text{ etc.} \\ 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \text{ etc.} \end{array}$$

and remarks that "*these exponents they call logarithms*, which are artificial numbers, so answering to the natural numbers, as that the addition and subduction of these answers to the multiplication and division of the natural numbers." And yet, Wallis does not come out, resolutely, with the modern definition of a logarithm, and use it.

A similar point of view was reached[5] by John Bernoulli I in a letter of May, 1694, addressed to Leibniz. Bernoulli discusses an "ideam novi $\cdots$ *calculi percurrentis*," a terminology later discarded in favor of calculus of "exponential quantities." He speaks of the construction of exponential curves $x^{x} = y$ by

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[1] C. I. Gerhardt, *Der Briefwechsel v. G. W. Leibniz mit Mathematikern*, Bd. I, Berlin, 1899, p. 230.

[2] G. Eneström, *Bibliotheca mathematica*, 3d S., Vol. 9, 1908–1909, p. 329.

[3] Ch. Reyneau, *Analyse demontrée*, Vol. I, Paris, 1708, p. xvii.

[4] *Analyse demontrée*, Vol. II, 1708, p. 806.

[5] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium philosophicum et mathematicum*, T. I, 1745, p. 8.

means of the ordinary logarithmic curve which he says is itself a curve of that type, having the equation $a^x = y$. Bernoulli assumes the logarithmic curve to be drawn and uses the graph for plotting the equation $x^x = y$. He assumes a value x_1, then measures off the ordinate log x_1 on the logarithmic curve and geometrically constructs the product $x_1 \log x_1 = \log y_1$. Finally he finds, again by the aid of the logarithmic curve, the antilogarithm y_1. This y_1, together with the assumed value of x_1, yields a point on the curve $x^x = y$. Thus the curve can be constructed by points. From our point of view, the interest of this process lies in the fact that the logarithmic curve and therefore the logarithm itself, is connected with the equation $a^x = y$. Here x is looked upon as the logarithm of y. Bernoulli makes no mention here of arithmetic and geometric progressions. His procedure involves the modern definition of a logarithm, which, however, he does not explicitly state. The process shows that Bernoulli passed from $x^x = y$ to $x \log x = \log y$, though he did not actually write down this last equation. In June, 1694, Leibniz sent J. Bernoulli a letter in reply, in which he writes[1] both $x^x = y$ and $x \log x = \log y$. We see from the above that Leibniz and J. Bernoulli had a grasp at this time of the exponential *function.*

II. From Leibniz and John Bernoulli I to Euler. 1712–1747.

UNSUCCESSFUL ATTEMPTS TO CREATE A THEORY OF LOGARITHMS OF NEGATIVE NUMBERS.

In the eighteenth century the tendency to take rules derived only for a special case and apply them to more general cases became more pronounced than it had been. It is a tendency which in the nineteenth century came to be called the "principle of the permanence of equivalent forms" or, better still, the "principle of the permanence of formal laws." To-day, we look upon these extensions as things we are at liberty to do or not to do, as we may please. If we find it most convenient in a given research to reject negative and complex numbers and confine ourselves to positive numbers, we may do so, but we are expected to state our position clearly, then to maintain it. In the seventeenth and eighteenth centuries it was not clearly felt that, logically, one had this freedom to extend or to limit the number concept. Negative numbers came to be used freely, yet this extension of the domain was done with misgivings, which show themselves in the names applied to them, such as *false* or *defective* numbers, *numeri ficti.* Still more pronounced was the feeling of discomfort toward bi or $a + bi$; such numbers were called *imaginary, impossible.* It was felt that the validity of negative and complex numbers should be *proved,* not *assumed;* that the rules of operation with such numbers was a matter requiring *demonstration.* Hence the eighteenth century mathematicians, including even men of the type of Laplace, tried to prove the rule of signs in the multiplication of two negative numbers. The "proofs" given were futile; they rested on a syllogism without

[1] Leibnitii et Bernoullii *Com. phil. et math.,* I, p. 10.

a major premise. This difference in the point of view must be borne in mind in the history of the extension of the logarithmic concept to negative and complex numbers. It will help to explain how it was that the controversy on this subject lasted for a whole century and reached well into the nineteenth century.

Several of the eighteenth century mathematicians of the first rank, particularly Euler, used imaginaries freely; some other mathematicians looked upon the imaginaries with suspicion. Here and there an eighteenth century mathematician declaims loudly against the use both of the negative and the imaginary.[1] Interesting is the language used by Leibniz in 1702. He speaks of the imaginary factors of $x^4 + a^4$ as "an elegant and wonderful recourse of divine intellect, an unnatural birth in the realm of thought, almost an amphibium between being and non-being."[2] Most wonderful was the result reached[3] in 1702 by John Bernoulli[4] (1667–1748). He explained the transformation of the differential $adz \div (b^2 + z^2)$ into $-adt \div 2bt\sqrt{-1}$, by means of the relation $z = (t-1)$. $b\sqrt{-1} \div (t+1)$ and thereby showed that the integral can be expressed as an arctangent and also as a logarithm. In this manner he pointed out a relation between the logarithm of an imaginary number and the arctangent. This *logarithme imaginaire*, as John Bernoulli called it, was so novel and so foreign to the thought of the time that it caused little comment.

In a letter of the same year (1702), dated June 24 and addressed to John Bernoulli, Leibniz speaks of imaginary logarithms in connection with the problems of integration.[5] There is danger of attaching too much importance to passages of this sort. To Leibniz and J. Bernoulli an imaginary often meant simply non-existence. If logarithms of imaginary numbers were believed to exist, nothing is here brought out as to the nature of such logarithms.

The controversy on logarithms which agitated mathematicians for more than a century did not originate primarily in discussions of imaginary number, but rather in discussions of negative number. Are negative numbers less than nothing? If they are, then in a proportion $1 : -1 = -1 : 1$, the greater number is to the less, as the less is to the greater — an impossibility. This matter was discussed by many writers, including Leibniz, Newton, D'Alembert, Maclaurin, Rolle and Wolf. Leibniz published a paper on this subject in 1712.[6] He considered the above proportion impossible in fact, but maintained that such proportions may be used with the same advantage and safety with which other inconceivable quantities are used. Leibniz said that a ratio may be considered imaginary, when it has no logarithm. The ratio $-1 \div 1$ has no logarithm; for, there would result $\log(-1/1) = \log(-1) - \log 1 = \log(-1)$.

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[1] See Cantor, *op. cit.*, Vol. 4, 1908, pp. 79–90.

[2] Leibniz, *Werke*, Ed. Gerhardt, 3. F., Bd. V, 1858, Berlin, p. 357: *Itaque elegans et mirabile effugium reperit in illo Analyseos miraculo, idealis mundi monstro, pene inter Ens et non-Ens Amphibio, quod radicem imaginariam appellamus.*

[3] Joh. Bernoulli, *Opera*, Vol. I, Laus. et Genevæ, 1742, p. 399.

[4] For the sake of distinction, this John Bernoulli is frequently designated as John Bernoulli I.

[5] *Leibnitii et J. Bernoullii Commerc. Phil. et math.*, T. II, 1745, p. 81.

[6] *Acta Eruditorum*, 1712, pp. 167–169; *Werke*, 3. F., Bd. V, Halle, 1858, pp. 387–389.

Leibniz declared that -1 has no real logarithm; such a logarithm could not be positive, for a positive logarithm corresponds to a number larger than 1; the logarithm could not be negative, for a negative logarithm corresponds to a positive number, less than unity. The only alternative remains, therefore, to declare the logarithm of -1 as not really true, but imaginary. He arrives at the same conclusion from the consideration that if there really existed a logarithm of -1, then half of it would be the logarithm of the imaginary number $\sqrt{-1}$, a conclusion which he considered absurd. We notice in these statements of Leibniz a double use of the term imaginary: (1) in the sense of non-existent, (2) in the sense of a number of the type $\sqrt{-1}$.

On March 16 of this year, before the appearance of the article, Leibniz mentioned the subject in a letter to John Bernoulli. That letter opened up a friendly controversy between the two men on the logarithms of negative and imaginary numbers. In their correspondence they debated this question for sixteen months. At that time they were the only ones interested in this question; in fact, they were the only ones to whom the problem of the existence or non-existence of logarithms of negative numbers had occurred. The controversy opens up a number of most interesting points and gives an insight into the algebraic concepts of the time as could not be obtained readily in any other way. For brevity let $+n$ or $+x$ indicate a "positive number," $-n$ or $-x$ a "negative number," $\log(+n)$ the "logarithm of a positive number," $\log(-n)$ the "logarithm of a negative number," i or in, an "imaginary number." The following is a synopsis of the correspondence:[1]

March 16, 1712. Leibniz to J. Bernoulli. This is the letter already referred to. L. says that $-1/1$ is imaginary, since it has no logarithm.

May 25, 1712. J. Bernoulli to Leibniz. B. rejects L.'s proof that the ratio $1 : -1$, or $-1 : 1$ is imaginary, for the reason that $-x$ has a logarithm. We have $dx : x = -dx : -x$; hence, by integration, $\log x = \log(-x)$. The logarithmic curve $y = \log x$ has therefore two branches, symmetrical to the axis Y, just as the hyperbola has two opposite branches.

June 30, 1712. Leibniz to J. Bernoulli: L. repeats his argument that $\log(-2)$ does not exist; for, if it did, its half would equal $\log\sqrt{-2}$, an impossibility. The rule for differentiating, $d \log x = dx : x$, does not apply[2] to $-x$. In the logarithmic curve $y = \log x$, x cannot decrease to 0 and then pass to the opposite side, since the curve cannot cut the Y axis, which is asymptotic to it.

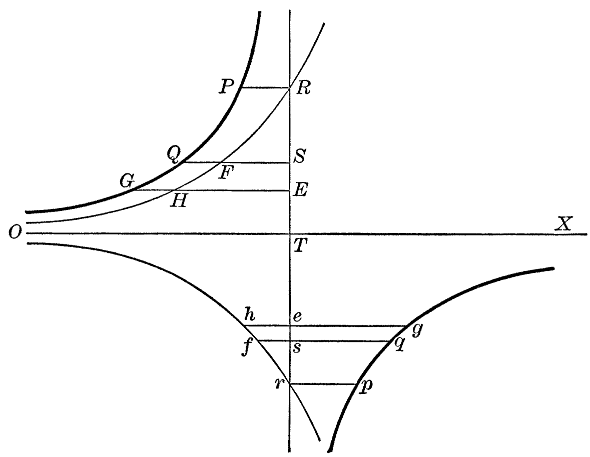
August 13, 1712. J. Bernoulli to Leibniz: The argument that $\log(-2)$ does not exist, because $\log\sqrt{-2}$ does not exist is invalid. I deny that $\log(\sqrt{-2})$ is half of $\log(-2)$, even though it be true that $\log\sqrt{2} = \frac{1}{2}\log 2$. The difference is that $\sqrt{2}$ is a mean proportional between 1 and 2; $\sqrt{-2}$ is not a mean proportional between -1 and -2. Just as $\log\sqrt{1 \times 2} = \frac{1}{2}\log 2$, so is $\log\sqrt{-1 \times -2} = \frac{1}{2}\log(-2)$. That is, $\log\sqrt{2} = \frac{1}{2}\log 2 = \frac{1}{2}\log(-2)$. In passing from $+x$ to $-x$ in a curve it is not necessary that the curve cut the y-axis. Witness the conjugate hyperbola whose abscissas are common to $+$ and $-$ ordinates, but never to vanishing ones. In $y = \log x$, one branch of the curve passes into the other at infinity, when $x = 0$, in the same manner as in the conchoid of Nicomedes and other curves.

[1] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium Philosophicum et Mathematicum*, Tomus Secundus, Lausannæ et Genevæ, 1745, pp. 269, 276, 278, 282, 287, 292, 296, 298, 303, 305, 312, 315.

[2] *Sed hæc regula, quod differentiale divisum per numerum dat differentiale Logarithmi, et quævis alia de Logarithmorum natura et constructione non habet locum in numeris negativis, ut reperies, ubi demonstrare voles.*

Sept. 18, 1712. Leibniz to J. Bernoulli: Logarithms are numbers in arithmetic progression, corresponding to numbers in geometric progression, of which one number may be 1 and another may be any positive number. Assume $\log 1 = 0$ and $\log 2 = 1$. In the geometric progression thus limited, $-n$ can never be obtained, no matter how many third proportionals are formed. In the series 1, 2, 4, the mean proportional between 1 and 4 is both $+2$ and -2. But -2 cannot be in the same geometric progression which contains $+2$; that is, no value of e makes $-2 = 2^e$ or $e = \log(-2)$; hence there is no logarithm of -2, and a curve $e = \log x$ that is satisfied by $x = 1$ and $x = 2$, cannot be satisfied by $x = -2$. *Otherwise thus:* If -2 has a logarithm, then the half of this logarithm exists and is the logarithm of $\sqrt{-2}$. But $\sqrt{-2}$ is an impossible number; hence, half of $\log(-2)$ is impossible, and the whole, or $\log(-2)$ is impossible. *Another point:* In logarithmic theory, n^e or $\sqrt[e]{n}$ is represented by $\log n \cdot e$, or $\log n : e$; nn or $n : n$ is represented by $\log n \pm \log n$; n is represented by $\log n$; by what is $-n$ represented? There is no mode of representation below the ones already named. *Again:* Granting for the moment that $\log(-2)$ exists, it follows that $\log \sqrt{-2}$ is half of it, for $\sqrt{-2}$ is the mean proportional between $+1$ and -2; hence, $\log \sqrt{-2} = (\log 1 + \log(-2)) : 2 = \frac{1}{2} \log(-2)$.

Nov. 9, 1712. J. Bernoulli to Leibniz: B. says that he sees nothing in the last letter which proves the impossibility of $\log(-n)$. He admits that there is no transition from a (geometric) series of positive terms to one of negative terms, and so $\log(-n)$ does not exist in this case.[1] But negative numbers determine their own peculiar series starting with -1, instead of $+1$ Thereby the same logarithmic properties follow for $-n$ as for $+n$. He reiterates that



$\log n = \log -n$. To show that $y = \log x$ has two branches, he uses the rectangular hyperbola $PQGpqg$ and lets SF and EH be proportional to the hyperbolic areas $RSQP$ and $REGP$. Let PR and GE be constants and FS a variable. As S touches T, FS is infinite and the area is infinite. Now keeping to the same law of generation of the curve RFH, let the point S proceed to e (for what can hinder this?). The area upon Re is partly + and partly −, and equal to EP, when $TE = Te$. We have then $EH = eh$. Similarly, if $Ts = TS$, then $sf = SF$. Thus arises the branch hfr which, with HFR, constitutes the one logarithmic curve, just as the two branches of the hyperbola constitute one curve. If $TR = +1$, $Tr = -1$, $TS = +n$, $Ts = -n$, then $SF = \log n$, $sf = \log(-n)$. As $SF = sf$, we must have $\log n = \log(-n)$.

Jan., 1713. Leibniz to J. Bernoulli: Assuming $2^e = x$, if $x = 1$ then $e = 0$, if $x = 2$ then $e = 1$. When $x = -1$, e cannot be assigned.

[1] *Hoc unum efficis omnibus Tuis argumentis, ut ostendas non dari transitum ex serie numerorum affirmativorum in seriem negativorum, hoc est, assumpta unitate* (*nempe* $+1$) *pro initio seriei numerorum, nullum numerum negativum ex illa serie inveniri posse, adeoque nullos eorum logarithmos hoc casu existere; quod quidem non nego. Sed hoc non impedit, quominus numeri negativi suam peculiarem constituant seriem, assumta pro eorum initio unitate negativa,* (*nempe* -1).

Feb. 28, 1713. J. Bernoulli to Leibniz: If in $v^e = x$ we assume $x = 2$ and $e = 1$, also $x = 1$ and $e = 0$, then truly e cannot be assigned when $x = -1$. As these assumptions are arbitrary, change them so that $x = -1$ when $e = 0$, then e can be assigned for any $-x$.

April 26, 1713. Leibniz to J. Bernoulli: You say that my values for e and x in $2^e = x$ are arbitrary. You let $x = -1$ and $e = 0$. Mine is the most natural. Aside from that consider *firstly* that we cannot have both $\log n$ and $\log(-n)$, for if $\log(-1) = 0$, then $\log(-1)^2 = \log 1 = 2 \times 0 = 0$, and $\log \sqrt{-1} = 0/2 = 0$. That is, one and the same logarithm is obtained for $+1$, -1, and i. *Secondly:* On your assumptions 2^0 has an infinite number of meanings, for 2^0 will equal -1, also $+1$, $\sqrt{-1}$, $\sqrt[4]{-1}$, $\sqrt[8]{-1}$, etc. Unless 2^0 is many-valued, these must all be equal to each other. If $2^0 = +1$, then 2^0 is single-valued and no such difficulty arises. *Thirdly:* If $x^e = -2$, $x^{2e} = +4$, but this transition from $-n$ to $+n$ you yourself reject. *Fourthly:* If $\log(-n)$ is real, then $\log \sqrt{-n}$ is real; hence, impossible numbers would have possible logarithms. The assumption that only $+n$ have logarithms avoids this trouble. *Fifthly:* From the beginning you have admitted that we cannot have $2^e = 1$ and $2^e = -1$ at the same time. But if you put $2^0 = -1$, then $(2^0)^2 = 2^0 = +1$. Hence $2^e = 1$ and $2^e = -1$, for $e = 0$. This is contrary to your admission.

All three things show that your hypothesis concerning $\log(-n)$ is unnatural, useless, and inadmissible. I have shown elsewhere that proportions cannot be formed, involving $-n$. If it be true that the two fractions $+1/-1$ and $-1/+1$ are equal, observe that fractions are not the same as ratios. It is evident from all this that the very foundations of things analytical have been neglected thus far.[1]

June 7, 1713. Bernoulli to Leibniz: What do you understand by *natural?* If that is natural which conforms with usage, then $\log(-n)$ is less natural than $\log(+n)$. The first of your five objections to $\log(-n)$ is that some $+n$, $-n$, in would have the same logarithms. I admit only that $\log(+n) = \log(-n)$. The half of any logarithm is not necessarily the logarithm of the square root; it is rather the logarithm of the mean proportional between $+1$ and $+n$, or -1 and $-n$. The mean proportional between -1 and -1 is $\sqrt{-1 \times -1} = +\sqrt{+1}$ or $-\sqrt{+1}$. There is nothing absurd in this. *Secondly:* I deny that $2^0 = \sqrt{-1} = \sqrt[4]{-1}$, etc. As just explained, $2^0 = \sqrt{-1 \times -1}$ and $= \sqrt{-1 \times -1 \times -1 \times -1}$, etc. All these radicals equal $\sqrt{+1}$ or ± 1. There is no discord in these results. *Thirdly:* You say that if $x^e = -2$, then $x^{2e} = +4$. The logarithmic curve shows this to be untrue. Twice $\log(-n)$ is not $\log n^2$. The third proportional of $-n$ is obtained from $-1 : -n = -n : x$. Hence if $x^e = -2$, then $x^{2e} = -2 \times -2 \div -1 = -4$. Consequently there is no crossing from $-n$ over to $+n$. *Fourthly:* My definition of the mean proportional of $-n$ does not lead to the absurd result that in has a possible logarithm. *Fifthly:* If $2^0 = -1$, then $2^{2 \cdot 0}$ is not $= +1$, but to $-1 \times -1 : -1 = -1$. Hence the absurd result does not follow that 2^0 is at the same time $+1$ and -1.

June 28, 1713. Leipniz to J. Bernoulli: I have no time to disprove your objections to my doctrine which makes $\log i$ impossible, the double of impossibles impossible, $\log n$ the double of $\log \sqrt{n}$. If you assume logarithms in which this is not so, that is nothing to me. I call the more *natural*, not that which is more customary, but that which is nearer to nature and the more simple.

July 29, 1713. J. Bernoulli to Leibniz: You do not deny that the assumption $+1$ is arbitrary, and that -1 is permissible. According to the latter, $\log(-1) = 0$. From this follows all I have previously said about $\log(-n)$.

It is easy to see that Leibniz and J. Bernoulli could not come to an agreement on $\log(-n)$, as long as they did not agree on the definition of "mean proportional" and "third proportional," when applied to negative numbers. There is no need on our part to enter into a minute discussion of the validity of the arguments presented. Bernoulli's argument involving infinite areas between the hyperbola and its asymptotes was repeatedly bombarded during the eighteenth century, but was never hit at its vulnerable point, namely the assumption

[1] *Ex quibus intelligitur, in ipsis rei Analyticæ fundamentis aliqua adhuc neglecta fuisse.*

that $\infty - \infty = 0$.[1] It is interesting to note that logarithms are part of the time connected with the two progressions, as in Napier's definition of a logarithm, but most of the time with the exponential concept as expressed in the exponential notation of the present time. Leibniz makes few appeals to geometry; J. Bernoulli uses curves repeatedly, as if more could be gotten out of a figure than is put into it.

Leibniz insists that $\log(-1)$ and $\log(\sqrt{-1n})$ do not exist. *Non potest dari Logarithmus* $\sqrt{-2}$. Non-existence is based on inconceivability. We shall see later that Euler puts a different intepretation upon Leibniz; he represents Leibniz as contending that $\log(-1)$ is imaginary, not as non-existing. Leibniz died three years after the close of this controversy. This correspondence between him and John Bernoulli during the years 1712 and 1713 was not published until 1745. Not until then did the logarithms of negative numbers engage the attention of mathematicians in general.

Meanwhile there are other researches demanding our attention. In an article in the *Philosophical Transactions* of London, published in 1714, Roger Cotes develops an important formula which in modern notation is

$$i\varphi = \log(\cos\varphi + i\sin\varphi).^2$$

In 1722, after the death of Cotes, this article was republished in his *Harmonia mensurarum*. He introduces as the "measure of a ratio" (mensura rationis) the logarithm of the ratio, multiplied by a constant or modulus. As Braunmühl points out, this "measure of a ratio" was long lost sight of, but was introduced anew in the nineteenth century. We have already called attention to the fact that Cotes himself was anticipated by Edmund Halley in this mode of measuring ratio.

Without stopping to explain how "the measure of the ratio" figures in Cotes's

[1] Proofs involving the comparison with each other of infinite areas in a plane appeal to our intuition with great force. They were accepted as valid by some writers of the nineteenth century as well as of the eighteenth century. Louis Bertrand based upon such comparison his proof of Euclid's parallel-postulate. See L. Bertrand, *Developpement nouveau de la partie elementaire des mathematiques*, T. II, a Geneve, 1778, pp. 19, 20. This proof was accepted by Johann Schultz. See J. Schultz, *Versuch einer genauen Theorie des Unendlichen*, Königsberg und Leipzig, 1788, pp. xi–xv. Similar reasoning was endorsed by Johann Heinrich Lambert. See Lambert, *Deutscher gelehrter Briefwechsel*, Bd. I, Berlin, 1781, p. 118, in a letter dated Feb. 2, 1766. Substantially Louis Bertrand's proof was published anonymously in Crelle's *Journal* in 1834. It was accepted as valid by A. De Morgan. See De Morgan on "Infinity and the Sign of Equality" in *Trans. Cambridge Phil. Society*, Vol. XI, Part I, p. 158. It was translated and published by W. W. Johnson in the *Analyst* (Des Moines, Vol. III, 1876, p. 103). See Cajori, *Teach. u. Hist. of Math. in U. S.*, 1890, p. 379.

[2] That Cotes derived this relation was pointed out by Timtschenko in his History of the Theory of Functions, 1899 [Russian], pp. 519–522. It receives emphasis also in an article by A. v. Braunmühl in *Bibliotheca mathematica*, 3d S., Vol. V, 1904, pp. 355–365. How it happened that so important a theorem should remain in the writings of Cotes for 185 years, without being detected, may perhaps be inferred from Cotes' mode of statement: "Nam sit quadrantis circuli quilibet arcus, radio CE descriptus, sinum habeat CX sinumque complemendi ad quadrantem XE; sumendo radium CE pro Modulo, arcus erit rationis inter $EX + XC\sqrt{-1}$ et CE mensura ducta in $\sqrt{-1}$."

derivation of his formula for $i\varphi$, his process may be roughly outlined in modern notation as follows: The area of the surface generated by an arc of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, when revolved around the Y-axis and taken between the limits $y = 0$, $y = y$, can be expressed in two ways, namely,

$$S = a\pi\left\{y\sqrt{1+\frac{a^2-b^2}{b^4}y^2}+\frac{b^2}{\sqrt{a^2-b^2}}\log\left(y\sqrt{\frac{a^2-b^2}{b^4}}+\sqrt{1+\frac{a^2-b^2}{b^4}y^2}\right)\right\},$$

$$S = a\pi\left\{y\sqrt{1-\frac{b^2-a^2}{b^4}y^2}+\frac{b^2}{\sqrt{b^2-a^2}}\varphi\right\},$$

where $\sin\varphi = y\sqrt{\dfrac{b^2-a^2}{b^4}} = iy\sqrt{\dfrac{a^2-b^2}{b^4}}$, $\cos\varphi = \sqrt{1+\dfrac{a^2-b^2}{b^4}y^2}$. Comparing the two expressions for S, we have

$$i\psi = \log(i\sin\psi + \cos\psi). \tag{1}$$

While Cotes was anticipated by John Bernoulli I in establishing a relation between the logarithm of an imaginary number and a goniometric function, Cotes, in turn, anticipated the continental mathematicians in the derivation of $i\varphi = \log(\cos\varphi + i\sin\varphi)$. It was much later, in a letter of Oct. 18, 1740, that Euler stated to John Bernoulli I that $y = 2\cos x$ and $y = e^{+x\sqrt{-1}} + e^{-x\sqrt{-1}}$ were both integrals of the differential equation $d^2y/dx^2 + y = 0$; that the two integrals were equal to each other, since both could be expanded into the same infinite series. Euler makes remarks from which it follows that he knew at that time the corresponding exponential expression for $\sin x$.[1] Both expressions are given by him in *Miscellanea Berolinensia* 1743 and again in his *Introductio in analysin*, Lausannæ, 1748, Vol. I, p. 104, where he gives also the all-important formulæ, $e^{+v\sqrt{-1}} = \cos v + \sqrt{-1}\sin v$, $e^{-v\sqrt{-1}} = \cos v - \sqrt{-1}\sin v$. From Cotes's formula $i\varphi = \log(\cos\psi + i\sin\psi)$, given in 1714 and 1722, to Euler's exponential form is an easy step, yet over a third of a century intervened between the first publication of the one and of the other. It is interesting to observe that neither Cotes nor Euler appear to hesitate in, or to recoil from, the use of $\log(\cos\psi + i\sin\psi)$, involving the logarithm of complex numbers. Moreover, neither Cotes, nor Euler in his *Introductio*, make any attempt to use this relation in the discussion of the theory of logarithms of complex numbers. Both were aware of the periodicity of the trigonometric functions. Had Cotes applied the idea of periodicity to $i\varphi = \log(\cos\psi + i\sin\psi)$ he might have anticipated Euler by many years in showing that the logarithm of a number has an infinite number of different values.

The theory of logarithms of negative numbers was incidentally touched by Euler very early, in his correspondence with John Bernoulli I. The letters which passed between these men in 1727–1731 have been in the possession of the Stockholm academy of sciences and have for the first time been published in full by

[1] *Bibliotheca mathematica*, 2d S., Vol. 11, 1897, pp. 48–49.

G. Eneström in 1902.[1] A translation from the original Latin into German was brought out by E. Lampe.[2] It will be remembered that Euler was a pupil of John Bernoulli I and had followed the two sons (Daniel and Nicolaus) of John Bernoulli to St. Petersburg. Euler was then 20 years old; John Bernoulli was 60. The following is a synopsis of the correspondence, in so far as it bears on logarithms:

Nov. 5, 1727. Euler to J. Bernoulli: The equation $y = (-1)^x$ is difficult to plot, since y is now positive, now negative, now imaginary. It cannot represent a continuous line.[3]

Jan. 9, 1728. J. Bernoulli to Euler: If $y = (-n)^x$, then $ly = xl\,(-n)$ and $dy/y = dx \cdot l(-n) = dx \cdot l(+n)$; for $dl(-z) = -dz/-z = +dz/+z = dlz$. Integrating, $ly = xln$, and $y = n^x$. Hence $y = (\pm 1)^x$ becomes $1^x = 1$, or $y = 1$.

Dec. 10, 1728. Euler to J. Bernoulli: I have arguments both for and against $lx = l(-x)$. If $lxx = z$, we have $\frac{1}{2}z = l\sqrt{xx}$. But $\sqrt{xx}$ is as much $-x$ as $+x$. Hence $\frac{1}{2}z = lx = l(-x)$. It may be objected that xx has two logarithms, but whoever claims two, ought to claim an infinite number.[4] Argument against: From the equality of the differentials we cannot infer the equality of the integrals. Moreover, $l(-x) = lx + l(-1)$; hence $l(-x) = lx$ only if $l(-1) = 0$. Again, if $lx = l(-x)$, then $x = -x$ and $\sqrt{-1} = 1$, but I rather think the conclusion from the equality of the logarithms to the equality of the numbers cannot be drawn. Your expression for the area of a circular sector[5] of radius a, viz. $aa/(4\sqrt{-1}) \times l(x+y\sqrt{-1})/(x-y\sqrt{-1})$, becomes for a quadrant, x being then 0, $aa/(4\sqrt{-1}) \cdot l(-1)$. Hence, if $l(-1) = 0$, we must have $\sqrt{-1} = 0$ and even $1 = 0$. Most celebrated Sir, what do you think of these contradictions?

April 18, 1729. J. Bernoulli to Euler: When I say that $lx = l - x$, it is to be understood that $l - (x)$ is meant, not $l(-x)$. Thus, $l - (x)^{\frac{1}{2}}$ is real, but $l(-x^{\frac{1}{2}})$ is imaginary. The area of the circular sector is 0, when $x = 0$, however much it ought to be equal to the quadrant. Let the constant Q be a quadrant, then we may write the area of the sector generally $= aa/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})+nQ$, so that, the first term vanishing when the sector becomes a quadrant, n may be so chosen as to make nQ any multiple or submultiple of the quadrant we need. For a semi-quadrant we have $aa/(4\sqrt{-1})l\sqrt{-1}$, which is 0, since $l\sqrt{-1} = 0$. Here we must take $n = \frac{1}{2}$.

May 16, 1729. Euler to J. Bernoulli: The difference between $l - (x)$ and $l(-x)$ is not clear to me. The expression $4a/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})$, thought to be constant, appears to me to be increasing, since $x = 0$ exhibits a vanishing sector. That nQ ought to be added, I do not as yet see. If n can be $\frac{1}{2}$, it can be $\frac{1}{4}$ and any number. It would be superfluous to show that $aa/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})$ must express a sector, if nQ alone were sufficient to represent any sector whatever. Neither of us can afford to run into paralogisms.

In this correspondence between John Bernoulli I and L. Euler, Bernoulli holds the view that $\log n = \log(-n)$. This is the same formula which he defended 16 years earlier, in letters to Leibniz. Now, as then, Bernoulli argues from the equality of two differentials to the equality of the two resulting general

[1] See *Bibliotheca mathematica*, 3d S., Vol. 4, 1902, pp. 344–388. Information on these letters is given also in the same journal, 2d S., Vol. 11, 1897, pp. 51–56; Vol. 13, 1899, p. 46.

[2] *Festschr.* z. Feier d. 200. Geburtstages L. Eulers, Leipzig and Berlin, 1907, pp. 119–137.

[3] In this synopsis we follow the notation used in the letters except in our use of slanting fractional lines and parentheses necessitated by them.

[4] *Posset quidem objici, xx habere duos logarithmos, sed hoc qui asser[ere vult] infinitos adjudicare deberet.*

[5] To facilitate the derivation of this expression, we note that

$$\int (a^2 - x^2)^{-\frac{1}{2}}\,dx = -i\int (x^2 - a^2)^{-\frac{1}{2}}\,dx = -i\log\left(x + \sqrt{x^2 - a^2}\right) + c$$

$$= -i/2\log(x+iy)^2/(x^2+y^2) - i/2\log a^2 + c = -i/2\log(x+iy)/(x-iy) - i/2\log a^2 + c.$$

integrals. In his argument on sectorial areas (Apr. 18, 1729) he confuses definite integrals with general integrals. His distinction between $l - (x)^{\frac{1}{2}}$ and $l(-x^{\frac{1}{2}})$ is not made clear. Perhaps he means simply that $l(-x^{\frac{1}{2}})$ shall signify $l(-x)^{\frac{1}{2}}$, when x itself is positive.

Euler's argument, that $e^{z/2} = \pm x$ yields $\log x = \log(-x)$, involves an interesting point. When we write $a^b = c$ and define $b = \log_a c$, a and c are taken to be both single-valued. Euler drops this restriction on c. He takes $e^{z/2} = \pm x$, so that the definition implied in his mode of procedure, viz. $z/2 = \log(\pm x)$, really amounts to two definitions, $z/2 = \log x$ and $z/2 = \log(-x)$. Now there is no objection, *a priori*, to two distinct definitions. In vector analysis we have at least two definitions of multiplication, yielding a vector-product $a \times b$, and a scalar product $a \cdot b$. The question to be considered is, can the two definitions be used side by side? Do the two fit together so as to give a non-contradictory logarithmic theory of complex numbers? The conclusion is drawn from the two definitions and the ordinary rules of operation, that all roots of $+1$ and -1, in other words, all complex numbers of unit modulus, have zero for their logarithm. This is certainly very simple, also quite useless. Previous to the Euler-Bernoulli correspondence no real contradictions had been pointed out in this theory. It was Euler who gave it a death blow by pointing out that $\log \sqrt{-1} = 0$ was in conflict with the formula $\sqrt{-1}\,\pi = 2l\sqrt{-1}$, resulting from J. Bernoulli's accepted expression for the area of a circular sector. The blow dealt by Euler was not considered fatal at the time. Definite integrals were as yet indefinitely comprehended. J. Bernoulli's theory of logarithms continued to find defenders. Euler made another important remark in his letter of December 10, 1728; he touches for the first time the truth that $\log n$ has an infinite number of values. But he does not pursue this matter further at this time.

THE UNION OF THE LOGARITHMIC AND EXPONENTIAL CONCEPTS.

The possibility of defining logarithms as exponents was recognized in the seventeenth century by John Wallis, but not until about 1742 do we find a systematic exposition of logarithms based on this idea. About this time it came to be recognized that involution has two inverses, different in kind, namely, evolution and logarithmation; in the first inverse we assume, in $a^b = c$, b and c as given and find a, in the second inverse we assume a and c as given and find b. Tropfke[1] names William Gardiner as the first to give the new definition of logarithms and to base the theory of logarithms upon it. It is given in the introduction to Gardiner's *Tables of Logarithms*, London, 1742. The definition is as follows: "The common logarithm of a number is the Index of that power of 10 which is equal to the number." It is practically certain that this definition is not due to Gardiner, but to William Jones. "The Explication of the Tables," says Gardiner, ". . . I have collected wholly from the papers of W. Jones, Esq." Maseres,[2] who reprinted this "Explication" in 1791, attributed it entirely to

[1] Tropfke, *op. cit.*, Vol. II, 1903, p. 142, note 576.
[2] Maseres, *Scriptores logarithmici*, Vol. II, London, 1791, p. 1.

Jones. Prof. W. W. Beman informs me that Jones's exposition of logarithms, as given in his *Synopsis palmariorum matheseos*, 1706, was based on Halley's treatment of 1695, but a posthumous paper by Jones, published in the *Philosophical Transactions* for the year 1771, gives the new definition. Whether Jones printed this definition before Gardiner is still undetermined. The one whose influence was greatest in emphasizing the new view was Euler, who, in his *Introductio*, 1748, Chap. VI, §102, gives the definition involving exponents. In this same chapter Euler gives an exposition of negative and fractional exponents and calls attention to the multiple values of a number having a fractional exponent, an explanation seldom found in mathematical treatises of that time. That the new definition of a logarithm was in every way a step in advance has been doubted by some writers. Certain it is that it involves internal difficulties of a serious nature.

[*To be continued.*]

MINIMUM COURSES IN ENGINEERING MATHEMATICS.

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Introduction.—This paper was suggested by a number of inquiries as to the nature and content of the course in engineering mathematics at the University of Colorado. This course is based on three entrance units in Mathematics, and consists of algebra, trigonometry, analytic geometry, the calculus, and least squares as prescribed courses and differential equations, higher calculus, vector analysis, Fourier's series and other advanced courses as electives.

The following outline[1] is not that of a complete course in engineering mathematics, nor even the average course, but, as the title of the paper indicates, the minimum course. The average course is based on this minimum but contains more material, both theoretical and applied. This outline gives an irreducible minimum. A course falling below this standard may be a good trade school course, it may be a most useful and practical course in many respects, but it is not a course in engineering mathematics.

Mathematics and Engineering.—Engineering mathematics is in no sense trade school mathematics or practical arithmetic. A trade school may have little use for mathematics as a science, but the engineering college demands a knowledge of principles as well as facts. This is particularly noticeable in the recent advances in the profession of civil engineering which have been along the lines laid down by Rankine and not by Trautwine. To the engineering student mathematics is as essential as anatomy is to the surgeon, as chemistry is to the apothecary, as drill is to the army officer. The professor of engineering is certainly on firm ground when he takes the stand that the mathematics taught to his students should not be too abstract on the one hand nor too concrete on the other. If the subject matter is too abstract it is unintelligible or uninteresting to the beginner; if it is

[1] The Editors, while in sympathy with the broad purposes of this paper, share no responsibility for the details of the suggested programs.