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NOTES ON GAUSS'S THEORIA MOTUS, SECTION 114.

By Mr. A. Hall, Jr., Washington, D. C.

In section 114 of the Theoria Motus we have the following expression:

\[
\begin{align*}
\frac{(0 \cdot 2 \cdot II) \delta + (O \cdot 2 \cdot II) D}{(0 \cdot 1 \cdot I) \delta + (O \cdot 1 \cdot I) D} \\
\times \frac{(1 \cdot 0 \cdot O) \delta' + (1 \cdot 0 \cdot O) D'}{(1 \cdot 2 \cdot II) \delta' + (1 \cdot 2 \cdot II) D'} \\
\times \frac{(2 \cdot 1 \cdot I) \delta'' + (2 \cdot 1 \cdot I) D''}{(2 \cdot 0 \cdot O) \delta'' + (2 \cdot 0 \cdot O) D''} = -1.
\end{align*}
\]

(7)

\(\delta, \delta', \delta''\) are curtate distances drawn from three positions of the earth to an object moving in a great circle about the sun; \(D, D', D''\) are curtate distances drawn from the sun to the earth. We take the origin of coordinates at the centre of the sun, and axes at the earth, wherever it may be, parallel to those at the sun. 

\((0 \cdot 1 \cdot 2)\) denotes

\[
\tan \beta \sin (a'' - a') + \tan \beta' \sin (a - a'') + \tan \beta'' \sin (a' - a),
\]

and to get \((O \cdot 1 \cdot 2)\) we replace \(a\) by \(L\), and \(\beta\) by \(B\), and so with the other expressions of \((7)\) in parentheses.

Now in this expression \((7)\) \(\delta\) cannot be zero, nor \(\delta', \delta'', D, D', D''\); therefore, if \((7)\) be indeterminate it must have a zero factor common to every one of its terms. It may be noticed that the condition given in the English edition of the Theoria Motus that \((7)\) should be indeterminate, although a correct condition as there given, is not the general condition, but a special case to which the general condition reduces when \(B = B' = B'' = 0\). Clearing the expression \((7)\) of fractions, we have:

\[
\begin{align*}
&\delta \delta' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \\
+ &D \delta' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \\
+ &DD \delta' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \times (A) \\
+ &DD' \delta' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \\
+ &DD'' \delta' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \\
+ &DD' \delta'' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \times (A) \\
+ &DD'' \delta' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \\
+ &DD' \delta'' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \times (A) \\
+ &DD'' \delta' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \\
+ &DD' \delta'' \delta'' \left[ (0 \cdot 2 \cdot II)(1 \cdot 0 \cdot O)(2 \cdot 1 \cdot I) + (0 \cdot 1 \cdot I)(1 \cdot 2 \cdot II)(2 \cdot 0 \cdot O) \right] \times (A) \\
&= 0.
\end{align*}
\]
By actually multiplying out the co-efficient of \( \partial^0 \partial'' \) we get for it

\[
\left\{ \begin{array}{l}
\tan \beta' \tan \beta'' \sin (L'' - L') \sin (L - \alpha') \\
+ \tan B' \tan B'' \sin (\alpha'' - \alpha') \sin (L - \alpha') \\
+ \tan \beta' \tan \beta \sin (L - L'') \sin (L' - \alpha') \\
+ \tan B'' \tan B \sin (\alpha - \alpha'') \sin (L' - \alpha') \\
- \tan \beta' \tan B'' \sin (\alpha'' - \alpha') \sin (L - \alpha') \\
- \tan B' \tan B \sin (\alpha - \alpha') \sin (L'' - \alpha'') \\
- \tan \beta' \tan B'' \sin (\alpha'' - \alpha') \sin (L - \alpha') \\
- \tan B'' \tan B \sin (\alpha - \alpha') \sin (L'' - \alpha'') \\
- \tan \beta \tan B' \sin (\alpha' - L) \sin (L'' - \alpha'') \\
- \tan \beta \tan B' \sin (\alpha' - L) \sin (L'' - \alpha'') \\
\end{array} \right. 
\]

By inspection of \((A)\) we see that if we take the co-efficient of \( \partial^0 \partial'' \), change 1 into 1, and I into 1 throughout, we will get the negative of the co-efficient of \( D' \partial^0 \partial'' \). But in \((B)\), which is identical with the co-efficient of \( \partial^0 \partial'' \), if we change 1 into 1, and I into 1, \((0.1.2)\) becomes \((0.1.2)\) and the sign of the second parenthesis is changed, so that, calling \( U \) the twelve term expression in the second parenthesis, the negative of the co-efficient of \( D' \partial^0 \partial'' \) becomes

\[
-(0.1.2) \cdot U,
\]

or the co-efficient of \( D' \partial^0 \partial'' \) becomes

\[
(0.1.2) \cdot U,
\]

so that the co-efficients of \( \partial^0 \partial'' \) and \( D' \partial^0 \partial'' \), each co-efficient being composed of two factors, have the common factor \( U \). The other factor of the co-efficient of \( D' \partial^0 \partial'' \) is derived from \((0.1.2)\) by substituting I for 1, corresponding to the introduction of \( D' \) into \( \partial^0 \partial'' \) in the place of \( \partial' \). In the same way, the co-efficient of \( D \partial^0 \partial'' \) is

\[
(0.1.2) \cdot U.
\]

Thus we get for \((A)\)

\[
[(0.1.2) + (0.1.2) + (0.1.2) + (0.1.2) + (0.1.2) + (0.1.2) + (0.1.2) + (0.1.2) + (0.1.2) + (0.1.2) + (0.1.2)] \cdot U = 0,
\]

which is the general form of equation \((7)\) when it is cleared of fractions. Now, the expression in the brackets cannot generally be zero, and hence the condition that \((D)\) or \((7)\) should be identically true is

\[
U = 0,
\]

and in this case \( \partial, \partial', \partial'' \) may have indeterminate values. In the special case
when \( B = B' = B'' = 0 \), the case which really occurs in practice, \( U \) is much simplified, and it is in this simpler form that the condition is given in the English translation.

In article 112 we have derived the equations

\[
\begin{align*}
(1) & \quad 0 = u (\delta \cos a + D \cos L) - n' (\delta' \cos a' + D' \cos L') + n'' (\delta'' \cos a'' + D'' \cos L''), \\
(2) & \quad 0 = u (\delta \sin a + D \sin L) - n' (\delta' \sin a' + D' \sin L') + n'' (\delta'' \sin a'' + D'' \sin L''), \\
(3) & \quad 0 = u (\delta \tan \beta + D \tan B) - n' (\delta' \tan \beta' + D' \tan B') + n'' (\delta'' \tan \beta'' + D'' \tan B'').
\end{align*}
\]

If we regard \( u, n', n'' \) as quantities to be determined, in order that they may have values which are not zero, we must have the determinant condition

\[
\begin{align*}
\delta \cos a + D \cos L, & \quad \delta'' \cos a'' + D'' \cos L'', \quad \delta' \cos a' + D' \cos L' \\
\delta \sin a + D \sin L, & \quad \delta'' \sin a'' + D'' \sin L'', \quad \delta' \sin a' + D' \sin L' = 0,
\end{align*}
\]

which is equation (8) of article 114. Multiplying out the determinant we have

\[
\begin{align*}
& (0 \cdot 1 \cdot 2) \delta \delta' \delta'' + (0 \cdot 1 \cdot 2) D \delta' \delta'' + (0 \cdot 1 \cdot 2) D' \delta \delta'' \\
& + (0 \cdot 1 \cdot 1) D'' \delta + (0 \cdot 1 \cdot 1) D' D'' \delta + (0 \cdot 1 \cdot 1) D' D'' \delta' \\
& + (0 \cdot 1 \cdot 2) D D' \delta'' + (0 \cdot 1 \cdot 2) D D' D'' = 0.
\end{align*}
\]

We see that equation (8) does not become indeterminate with (7) and may be used when (7) fails.

In article 114 we have given the following equations:

\[
\begin{align*}
(9) & \quad 0 = u [ (0 \cdot 1 \cdot 2) \delta + (0 \cdot 1 \cdot 2) D ] - n' [(0 \cdot 1 \cdot 2) \delta' + (0 \cdot 1 \cdot 2) D'] + n'' [(0 \cdot 1 \cdot 2) \delta'' + (0 \cdot 1 \cdot 2) D''], \\
(10) & \quad 0 = u (0 \cdot 0 \cdot 2) D - n' [(0 \cdot 1 \cdot 2) \delta' + (0 \cdot 1 \cdot 2) D'] + n'' [(0 \cdot 1 \cdot 2) \delta'' + (0 \cdot 1 \cdot 2) D''], \\
(11) & \quad 0 = u (0 \cdot 1 \cdot 0) D - n' [(0 \cdot 1 \cdot 1) D' + n'' [(0 \cdot 1 \cdot 2) \delta'' + (0 \cdot 1 \cdot 2) D'']].
\end{align*}
\]

Suppose \( (0 \cdot 1 \cdot 2) \) equals zero, then we cannot get from these equations values of \( \delta, \delta', \delta'' \). Now, if we draw from the sun radii vectores parallel to the radii vectores drawn from the three positions of the earth to the planet, the condition

\[
(0 \cdot 1 \cdot 2) = 0
\]

means that the three radii vectores so drawn from the sun lie in a plane. Suppose we take this plane for the plane of \( x, y \) in the system of co-ordinate axes at the sun. Then \( \beta = \beta' = \beta'' = 0 \), and we have for equation (9), remembering that \((0 \cdot 1 \cdot 2) = 0\),

\[
u Z - n' Z' + n'' Z'' = 0.
\]

Also (10) and (11) reduce to this same equation. And since \( Z = z, Z' = z', Z'' = z'' \),
from the way in which we have chosen our axes, (9), (10), and (11) each reduce to
\[ nZ = n'Z' = n''Z'' = 0, \tag{E} \]
which is a known, necessary relation (see article 112), and does not depend upon
the position of the earth at all. And merely by transformation of co-ordinates
(9), (10), and (11) must be derived from (E).

We have also derived in article 114 the following equations:

\begin{align*}
(4) \quad & o = n' [\{(0.2.1) \cdot \delta + (1.2.1) \cdot D\} - n'' [\{(1.2.1) \cdot \delta' + (1.2.1) \cdot D'\}]
(5) \quad & o = n' [\{(0.1.1) \cdot \delta + (1.1.1) \cdot D\} + n'' [\{(2.1.1) \cdot \delta'' + (1.1.1) \cdot D''\}]
(6) \quad & o = n' [\{(1.0.0) \cdot \delta' + (1.0.0) \cdot D'\} - n'' [\{(2.0.0) \cdot \delta'' + (1.0.0) \cdot D''\}]
\end{align*}

Likewise here, we come back merely to necessary relations if in these equations
a co-efficient of \( \delta \), or \( \delta' \), or of \( \delta'' \) becomes equal to zero. Thus, suppose in (4)
the co-efficient of \( \delta \), \( (0.2.1) = o \). Then we have lying in a plane the radius
vector drawn from the sun to the third position of the earth, the line drawn from
the sun parallel to the radius vector drawn from the first position of the earth to
the planet, and the line drawn from the sun parallel to the radius vector drawn
from the third position of the earth to the planet. Take the plane containing
these three lines as the plane of \( x, y \) in the system of co-ordinate axes at
the sun. Then we have \( \beta = \beta'' = B'' = o \), and equation (4) reduces to
\[ nZ - n'Z' - n''Z'' = 0; \]
also, \( z = Z, z'' = Z'' = o \), hence this equation reduces to
\[ nZ - n'Z' - n''Z'' = 0 \tag{F} \]

Now, we have as a necessary relation
\[ nZ = n'Z' + n''Z'' = 0 \]
or since \( z'' = o \).
\[ nZ = n'Z' = 0 \]
Hence we have from (F) \( Z' = o \), or the second position of the earth as well as
the third lies in the plane of \( x, y \), therefore the plane of \( x, y \) passing through the
sun and two positions of the earth is the orbit of the earth. Hence \( z = Z = o \),
also since \( z'' = o \), we have from the necessary relation
\[ nZ - n'Z' + n''Z'' = 0 \]
\( z' = o \), or the plane of the orbit of the planet coincides with the ecliptic. We
get the same result if we take any other co-efficient of \( \delta \), or \( \delta' \), or of \( \delta'' \) equal to
zero.

The formulae of this article are important, since they give the relations
between the distances of a planet from the earth at the times of three observa-
tions, and it is on the determination of these distances that a knowledge of the
orbit of the planet depends. Equation (8) shows that if two of the distances are known the third can be computed. Equation (5) furnishes the relation between the curtate distances used by Olbers in his well-known method of computing the orbit of a comet.

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SOLUTIONS OF EXERCISES.

10

Required the length of a thread wrapped spirally round the frustum of a given cone, the distance between the spires along the slant height being constant.

\[ A. \ B. \ Nelson. \]

Solution.

The development of the thread on a plane obtained by rolling the cone on the plane will be a spiral of Archimedes, the length of which is well known.

\[ DeVolson \ Wood. \]

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The result

\[
\frac{p^2q^2 + 4p^3r - 8q^3 + 2pqr + qr^3}{(r - pq)^2}
\]

is given as the equivalent of the function

\[
\left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2 + \left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)^2 + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2,
\]

where \(\alpha, \beta, \gamma\) are the roots of the cubic

\[ x^3 + px^2 + qx + r = 0. \]

Is this result correct?

\[ A. \ Hall. \]

Solution.

The result is not correct. The symmetric function \(\Sigma \left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2\) of the roots \(\alpha, \beta, \gamma\) of the cubic \(x^3 + px^2 + qx + r = 0\), expressed in terms of the co-efficients, is

\[
= \frac{-3p^2q^2 + 4p^3r + 4q^3 + 2pqr + qr^2}{(r - pq)^2}.
\]